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Transition in the Floquet Rates of a Driven Stochastic System

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Floquet theory is used to solve the Smoluchowski equation for a time-periodic system whose underlying dynamics exhibits a transition to deterministic chaos. For the stochastic version of this system, an abrupt transition occurs in the Floquet decay rates as parameters of the system are varied, leading to a much more rapid decay to the stationary state.

KEY WORDS: Fokker–Planck equation; Smoluchowski equation; Brownian motion in a potential well; Floquet theory; nonlinear response; driven Brownian particle.

1. INTRODUCTION

It is a great pleasure to contribute to this volume honoring Nico van Kampen, who has probably done more than anyone to bring clarity to the field of stochastic physics. In this paper, I consider a subject which has been a recurrent theme in his work, namely the response of a nonlinear system to a dynamic external field. This problem has become especially interesting because we now know that a nonlinear system coupled to a dynamic field will generally become chaotic as the parameters of the system are varied. A problem that has been little studied but is of growing interest is the behavior of a stochastic system whose underlying dynamics undergoes a transition to chaos. In this paper I consider such a system.

The problem I consider is that of a Brownian particle of mass m and radius R confined to an infinitely deep square-well potential with potential energy $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise. The square well is

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filled with a fluid with shear viscosity η and the particle is driven by a monochromatic external force $f(t) = \varepsilon \sin(2\pi f t)$, where ε is the amplitude of the force and f is its frequency. The Langevin equation for the particle *inside the well* is

$$dv/dt = -\beta v + \varepsilon \sin(2\pi f t) - F(t) \tag{1.1}$$

where β is the Stokes friction, $\beta = 6\pi R\eta/m$, and $F(t)$ is a delta-correlated white noise due to the many degrees of freedom of the fluid. [Note that $\langle F(t)\, F(t')\rangle = (k_B T/\beta)\, \delta(t-t')$, where k_B is Boltzmann's constant and T is the temperature of the fluid. Hydrodynamic memory is neglected.] If no fluid is present, the classical mechanical version of this system[1] undergoes a transition to deterministic chaos in certain regions of the phase space as parameters of the external field are varied.

In this paper, I study the behavior of this stochastic system in the approximation where the friction is very strong so that the velocity relaxes to equilibrium on a time scale short compared to the period of the external field. The behavior of the system is then described by the Smoluchowski equation.[2,3] In Section 2, I write the Smoloukowski equation for the driven system. In Section 3, I use Floquet theory to determine the time evolution of the system, and in Section 4, I obtain the Floquet decay rates of the system.

2. DRIVEN PARTICLE IN AN INFINITE SQUARE WELL

Let us first consider the Smoluchowski equation for a particle confined to an infinitely deep square-well potential in the presence of white noise. The Smoluchowski equation for the particle in the interval $0 < x < L$ is

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2} \tag{2.1}$$

where $P(x, t)$ is the probability density of finding the particle at point x at time t, the diffusion coefficient $D = k_B T/m\beta$, and the boundary conditions are

$$\left.\frac{\partial P}{\partial x}\right|_{x=0,L} = 0$$

to ensure that no probability flows through the walls. The solution to Eq. (2.1) takes the form

$$P(x, t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n(x) \tag{2.2}$$

with the states $\phi_0(x) = 1/L$ and $\phi_n(x) = (\sqrt{2}/L)\cos(n\pi x/L)$ (for $n \neq 0$) for probability normalized to 1 on the interval $0 < x < L$. The eigenvalues are $\lambda_n = n^2\pi^2 D/L^2$. The coefficients c_n are determined by the initial conditions and are defined by

$$c_n = \int_0^L dx \frac{\phi_n(x)}{\phi_0(x)} P(x, 0) \tag{2.3}$$

with $c_0 = 1$.

Let us now drive the particle with a monochromatic external field. The Smoluchowski equation for the driven system takes the form

$$\frac{\partial P(x, t)}{\partial t} = \frac{\varepsilon}{\beta} \sin(2\pi f t) \frac{\partial P(x, t)}{\partial x} + D \frac{\partial^2 P(x, t)}{\partial x^2} \tag{2.4}$$

with the same boundary conditions as for Eq. (2.1). If we assume a solution of the form

$$P(x, t) = \sum_{m=0}^{\infty} c_m(t)\, \phi_m(x) \tag{2.5}$$

then we find that the coefficients $c_m(t)$ obey the equation

$$\frac{\partial c_m(t)}{\partial t} = -\frac{m^2\pi^2 D}{L^2} c_m(t) - \frac{2\varepsilon}{\beta L} \sin(2\pi f t) \sum_{\substack{n=1 \\ (n \neq m)}}^{\infty} a_{m,n}\, c_n(t) \tag{2.6}$$

where n is chosen so that $(n \pm m)$ is odd and

$$a_{m,n} = \left(\frac{n}{n-m} + \frac{n}{n+m}\right) \tag{2.7}$$

It is convenient to write Eqs. (2.4) and (2.6) in dimensionless form. Let us introduce angle $\theta = \pi x/L$, time $\tau = \pi^2 D t/L^2$, frequency $f_0 = L^2 f/\pi^2 D$, and coupling constant $q = 2\varepsilon L/\beta\pi^2 D$. Then Eq. (2.4) can be written

$$\frac{\partial P(\theta, \tau)}{\partial \tau} = \frac{\pi}{2} q \sin(2\pi f_0 \tau) \frac{\partial P}{\partial \theta} + \frac{\partial^2 P}{\partial \theta^2} \tag{2.8}$$

and Eq. (2.6) becomes

$$\frac{\partial c_m(\tau)}{\partial \tau} = -m^2 c_m(\tau) - q \sin(2\pi f_0 \tau) \sum_{\substack{n=1 \\ (n \pm m)\ \text{odd}}}^{\infty} a_{m,n}\, c_n(\tau) \tag{2.9}$$

Equation (2.9) gives the evolution of the system in terms of a set of first-order differential equations with time-periodic coefficients, so we can use Floquet theory to determine their future evolution.

3. FLOQUET THEORY

There has been considerable work using Floquet theory to describe the time evolution of quantum systems with time-periodic Hamiltonians.[4] In this section I use Floquet theory to study the evolution of a Fokker–Planck equation (Smoluchowski equation) with time-periodic coefficients. Let us write Eq. (2.9) in the abstract form

$$\frac{\partial}{\partial \tau} |c(\tau)\rangle = \bar{\bar{W}}(\tau)\, |c(\tau)\rangle \tag{3.1}$$

where $\bar{\bar{W}}(\tau) = \bar{\bar{W}}(\tau + T_0)$, $T_0 = 1/f_0$, $\langle n | c(\tau)\rangle \equiv c_n(\tau)$, and matrix elements are given by

$$\langle m|\, \bar{\bar{W}}(\tau)\, |n\rangle = -n^2 \delta_{m,n} - q \sin(2\pi f_0 \tau) a_{m,n} \tag{3.2}$$

Let us assume that Eq. (3.1) has Floquet solutions of the form

$$|c(\tau)\rangle = e^{\Lambda_\alpha \tau}\, |\chi_\alpha(\tau)\rangle \tag{3.3}$$

where $|\chi_\alpha(\tau)\rangle = |\chi_\alpha(\tau + T_0)\rangle$, and Λ_α are the Floquet decay rates of the system. It is easy to show that

$$\bar{\bar{\mathscr{W}}}(\tau)\, |\chi_\alpha(\tau)\rangle = \Lambda_\alpha\, |\chi_\alpha(\tau)\rangle \tag{3.4}$$

where

$$\bar{\bar{\mathscr{W}}}(\tau) = \bar{\bar{W}}(\tau) - \partial/\partial\tau \tag{3.5}$$

These states $|\chi_\alpha(\tau)\rangle$ are right eigenvectors of the operator $\bar{\bar{\mathscr{W}}}(\tau)$ with eigenvalues Λ_α. The operator $\bar{\bar{\mathscr{W}}}(\tau)$ is not self-adjoint. Thus the left and right eigenvectors will not be the same. Let us introduce left eigenvectors $\langle \psi_\alpha(\tau)|$ of the operator $\bar{\bar{\mathscr{W}}}(\tau)$

$$\langle \psi_\alpha(\tau)|\, \bar{\bar{\mathscr{W}}}(\tau) = \langle \psi_\alpha(\tau)|\, \Lambda_\alpha \tag{3.6}$$

It is straightforward to show that $\langle \psi_\alpha(\tau) | \chi_\alpha(\tau)\rangle = 0$ if $\Lambda_\alpha \neq \Lambda_\beta$. We shall always normalize these states so that $\langle \psi_\alpha(\tau) | \chi_\beta(\tau)\rangle = \delta_{\alpha,\beta}$. Furthermore, completeness requires that

$$\sum_\alpha |\chi_\alpha(\tau)\rangle\langle \psi_\alpha(\tau)| = 1$$

We shall assume that the eigenvectors of $\bar{\bar{\mathscr{W}}}(\tau)$ form a complete set.

Let us now expand $|c(\tau)\rangle$ in terms of eigenstates $|\chi_\alpha(\tau)\rangle$. Thus we write

$$|c(\tau)\rangle = \sum_\alpha A_\alpha e^{\Lambda_\alpha \tau} |\chi_\alpha(\tau)\rangle \tag{3.7}$$

The coefficients A_α are determined in terms of initial conditions $|c(0)\rangle$,

$$A_\alpha = \langle \psi_\alpha(0) | c(0)\rangle$$

Thus

$$|c(\tau)\rangle = \sum_\alpha e^{\Lambda_\alpha \tau} |\chi_\alpha(\tau)\rangle \langle \psi_\alpha(0) | c(0)\rangle \tag{3.8}$$

If we use the periodicity of $|\chi_\alpha(\tau)\rangle$, we find that after one period, T_0 of the external field

$$|c(T_0)\rangle = \sum_\alpha e^{\Lambda_\alpha T_0} |\chi_\alpha(0)\rangle \langle \psi_\alpha(0) | c(0)\rangle \tag{3.9}$$

or, in terms of states $c_n(T)$, we can write

$$c_n(T_0) = \sum_m U_{nm}(T_0)\, c_m(0) \tag{3.10}$$

where

$$U_{nm}(T_0) = \sum_\alpha e^{\Lambda_\alpha T_0} \langle n | \chi_\alpha(0)\rangle \langle \psi_\alpha(0) | m\rangle \tag{3.11}$$

The matrix $U_{nm}(T_0)$ can be constructed from the solution to Eq. (2.8) after one period of the external field. From its eigenvalues, we can find Λ_α and from its left and right eigenvectors we obtain $\langle n | \chi_\alpha(0)\rangle$ and $\langle \psi_\alpha(0) | m\rangle$.

4. NUMERICAL DETERMINATION OF FLOQUET RATES

The Floquet decay rates will be a function only of the dimensionless variables q and f_0 and are complex numbers. I have computed them for several values of f_0 and a range of q. For all values of q and f_0 consided one always finds one Floquet rate (which is designated Λ_0) equal to zero. Thus, this system does have a stationary state. Furthermore, for $q \ll 2\pi f_0$ the lowest Floquet rates behave as $\Lambda_\alpha \approx \alpha^2 + i0$. However, at $q \approx 2\pi f_0$ there is an abrupt transition in the Floquet rates. The lowest nonzero rate goes from $\Lambda_1 = -1 + i0$ to $\Lambda_1 \approx -\Gamma + i0$, where $\Gamma = \Gamma(f_0, q)$. (My numerical work did not have sufficient accuracy to make it possible to obtain a good

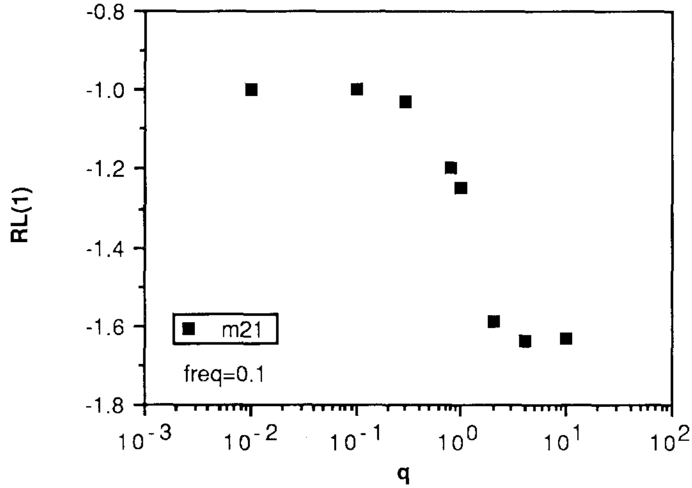

Fig. 1. A plot of $\mathrm{RL}(1)=\mathrm{Re}\,\Lambda_1$ versus q for $f_0=0.1$ and case m21.

estimate for Γ.) Thus, there is an abrupt increase in the rate of decay of the system to its stationary state. These results are shown in Figs. 1–4. To obtain these figures, I have integrated Eqs. (2.6) for systems of two different sizes. In one case I kept only the first 21 equations $(0 \leqslant n \leqslant 20)$, yielding a matrix $U_{n,m}(T_0)$ with 21×21 elements. In the other case I kept the first 61 equations $(0 \leqslant n \leqslant 60)$ yielding a matrix $U_{n,m}(T_0)$ with 61×61 elements. I denote these two cases m21 and m61, respectively.

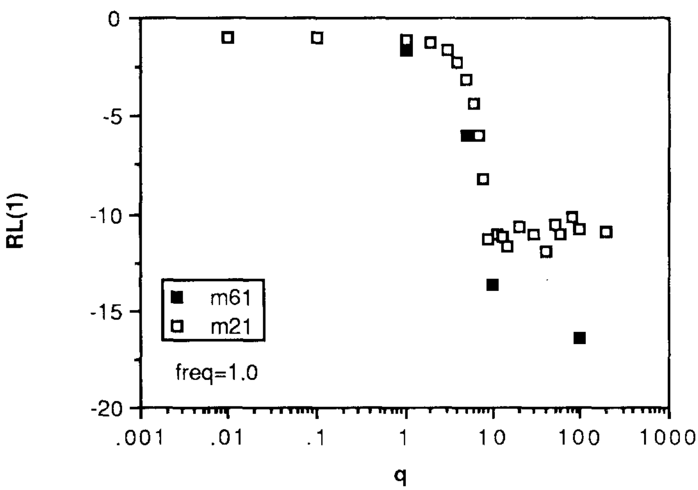

Fig. 2. A plot of $\mathrm{RL}(1)=\mathrm{Re}\,\Lambda_1$ versus q $f_0=1.0$ for cases (□) m21 and (■) m61.

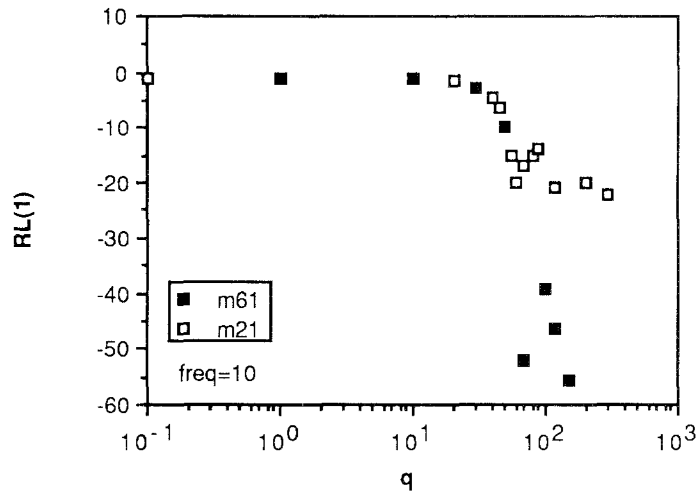

Fig. 3. A plot of $RL(1) = \mathrm{Re}\, \Lambda_1$ versus q for $f_0 = 10$ for cases (□) m21 and (■) m61.

In Fig. 1, I plot the real part of the lowest Floquet rate Λ_1 as a function of q for frequency $f_0 = 0.1$ and case m21. A transition in the lowest decay rate occurs at $q_c \approx 2\pi(0.1)$. The real part of Λ_1 drops from -1.0 to -1.6 to -1.7 and levels out there. In Fig. 2, I plot $\mathrm{Re}\, \Lambda_1$ as a function of q for frequency $f_0 = 1.0$ for cases m21 and m61. The transition now occurs at $q_c \approx 2\pi(1.0)$. For $q < q_c$ and at the transition point itself, the eigenvalue Λ_1 is insensitive to the size of the matrix used. However, for $q > q_c$, its value appears to be limited by the size of the matrix. In Fig. 3, I plot $\mathrm{Re}\, \Lambda_1$

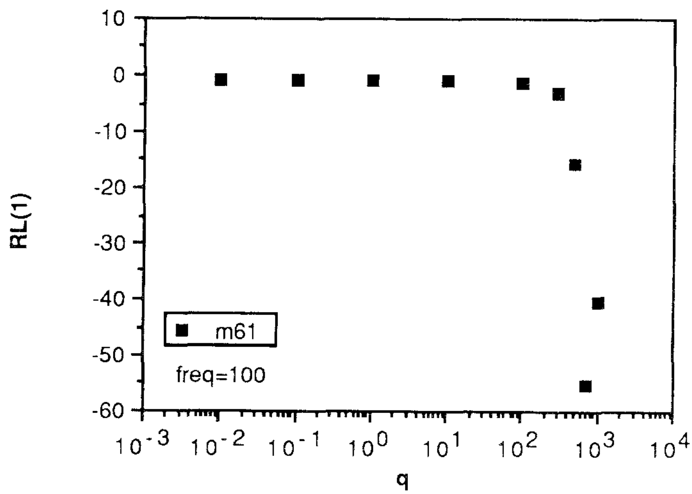

Fig. 4. A plot of $RL(1) = \mathrm{Re}\, \Lambda_1$ versus q for $f_0 = 100$ for case m61.

versus q for frequency $f_0 = 10$ for cases m21 and m61. Again the eigenvalue is relatively insensitive to the size of the matrix $U_{nm}(T_0)$ for $q \leqslant 2\pi f_0$, but for $q > 2\pi f_0$ it drops to a value which appears to be determined by the size of the matrix. Finally, in Fig. 4 I plot Re Λ_1 for frequency $f_0 = 100$ and case m61. Again a transition occurs at $q \approx 2\pi f_0$.

Since we are so far from the conservative dynamic regime, it is not clear whether or one this transition is directly related to the underlying transition to chaos. But the possiblity if a connection is intriguing. I hope to say more about this is a subsequent paper.

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